

# Numerably Contractible Spaces

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## Abstract

Numerably contractible spaces play an important role in the theory of homotopy pushouts and pullbacks. The corresponding results imply that a number of well known weak homotopy equivalences are genuine ones if numerably contractible spaces are involved. In this paper we give a first systematic investigation of numerably contractible spaces. We list the elementary properties of the category of these spaces. We then study simplicial objects in this category. In particular, we show that the topological realization functor preserves fibration sequences if the base is path-connected and numerably contractible in each dimension. Consequently, the loop space functor commutes with realization up to homotopy. Finally, we give simple conditions which assure that free algebras over a topological operad are numerably contractible.

## 1 Introduction

A *numerably contractible space* is a topological space  $X$  which admits a numerable cover by sets  $U \subset X$  for which the inclusions are nullhomotopic. Numerably contractible spaces are of importance in homotopy theory, as we will explain.

Some important weak homotopy equivalences are strict ones if the spaces involved are numerably contractible. Let  $k\text{Top}^*$  denote the category of based  $k$ -spaces. For  $X$  in  $k\text{Top}^*$  let  $JX$  denote the James construction on  $X$  in  $k\text{Top}^*$ , i.e. the based free topological monoid on  $X$ . In [4, (17.3)] and [13, Cor. 3.4] D. Puppe proved:

**1.1 Theorem:** If  $X$  is  $h$ -wellpointed, path-connected and numerably contractible, then  $JX \simeq \Omega\Sigma X$ .

For  $X$  in  $k\text{Top}^*$  let  $\mathbb{C}_n^*(X)$  denote the based free algebra over the operad  $\mathcal{C}_n$  of little  $n$ -cubes. P. May constructed a weak equivalence  $\mathbb{C}_n^*(X) \rightarrow \Omega^n \Sigma^n X$  for a path-connected  $X$  [10]. In his thesis H. Meiwes proved [11].

**1.2 Theorem:** If  $X$  is as in Theorem 1.1, then May's map  $\mathbb{C}_n^*(X) \rightarrow \Omega^n \Sigma^n X$  is a genuine homotopy equivalence.

In the context of these theorems we share D. Puppe's point of view [13]: "Frequently a weak homotopy equivalence is considered as good as a genuine one, because for spaces having the homotopy type of a  $CW$ -complex there is no difference and most interesting spaces in algebraic topology are of that kind. I am not going to argue against this because I agree with it, but I do think that the methods by which we establish the genuine homotopy equivalences give some new insight into homotopy theory."

Indeed, constructing homotopy equivalences between spaces which are not necessarily of the homotopy type of  $CW$ -complexes deprives one of the algebraic side of homotopy theory, so that these constructions have a different, more geometric flavor.

We do not know whether A. Dold introduced the notion of a numerably contractible space, but he was certainly among the first ones to work with them. Following J. Smrekar [15], we therefore also call such a space a Dold space. In his paper "Partitions of Unity in the Theory of Fibrations" Dold proved [5, Thm. 6.3]:

**1.3 Theorem:** Given a commutative diagram

$$\begin{array}{ccc} E & \xrightarrow{f} & E' \\ & \searrow p & \swarrow p' \\ & B & \end{array}$$

such that  $p$  and  $p'$  have the weak covering homotopy property and  $B$  is a Dold space then  $f$  is a fiberwise homotopy equivalence iff its restriction to every fiber is a homotopy equivalence.

As simple consequences of this result one has the following strengthened versions of well-known results about homotopy pullbacks (for simplicity we state the results for commutative squares. They also hold for homotopy commutative squares with a specified homotopy.)

**1.4 Proposition:** Let

$$\begin{array}{ccc} X_1 & \xrightarrow{u} & Y_1 \\ f \downarrow & & \downarrow g \\ X_0 & \xrightarrow{v} & Y_0 \end{array}$$

be a homotopy pullback. If  $v$  is a homotopy equivalence, so is  $u$ . Conversely, if  $u$  is a homotopy equivalence,  $g$  induces a surjection of sets of path-components, and  $Y_0$  is a Dold space, then  $v$  is a homotopy equivalence.

**1.5 Proposition:** Given a commutative diagram,

$$\begin{array}{ccccc} X_2 & \xrightarrow{f'} & X_1 & \xrightarrow{f} & X_0 \\ \downarrow w & \text{I} & \downarrow v & \text{II} & \downarrow u \\ Y_2 & \xrightarrow{g'} & Y_1 & \xrightarrow{g} & Y_0 \end{array}$$

- (1) suppose that II is a homotopy pullback. Then I is a homotopy pullback iff the combined square I+II is a homotopy pullback
- (2) suppose that I and I+II are homotopy pullbacks, that  $g'$  induces a surjection of sets of path-components and  $Y_1$  is a Dold space, then II is a homotopy pullback.

**1.6 Proposition:** Let

$$\begin{array}{ccc} X_1 & \xrightarrow{f} & X_0 \\ u \downarrow & & \downarrow v \\ Y_1 & \xrightarrow{g} & Y_0 \end{array}$$

be a commutative square and  $F(f, x)$  the homotopy fiber of  $f$  over  $x \in X_0$ .

- (1) If the square is a homotopy pullback, then the induced map

$$F(f, x) \rightarrow F(g, v(x))$$

is a homotopy equivalence for each  $x \in X_0$ .

- (2) If for each  $x \in X_0$  the map  $F(f, x) \rightarrow F(g, v(x))$  is a homotopy equivalence and  $X_0$  is a Dold space, the square is a homotopy pullback.

We also have the following improved version of M. Mather's second cube theorem [9].

**1.7 Proposition:** Given a commutative cube diagram whose vertical faces are homotopy pullback,

$$\begin{array}{ccccc}
 A_0 & \xrightarrow{\quad} & A_1 & & \\
 \downarrow & \searrow & \downarrow & \searrow & \\
 & A_2 & \xrightarrow{\quad} & A_3 & \\
 B_0 & \xrightarrow{\quad} & B_1 & \xrightarrow{\quad} & B_3 \\
 \searrow & \downarrow & \searrow & \downarrow f & \\
 & B_2 & \xrightarrow{\quad} & B_3 & 
 \end{array}$$

then

- (1) the top face is a homotopy pushout if the bottom face is a homotopy pushout.
- (2) the bottom face is a homotopy pushout if the top face is a homotopy pushout,  $f$  induces a surjection on path-components, and  $B_3$  is a Dold space.

Homotopy pushouts and pullbacks have become increasingly important tools in homotopy theory and homological algebra. E.g. there exist comparatively simple proofs of Theorem 1.1 based on Propositions 1.4 to 1.6 (unpublished). Another example is the following result of G. Allaud [1], which is an immediate consequence of Propositions 1.4 and 1.5.

**1.8 Proposition:** Let  $f : X \rightarrow Y$  be a based map of path-connected Dold spaces such that  $\Omega f : \Omega X \rightarrow \Omega Y$  is a homotopy equivalence. Then  $f$  is a homotopy equivalence.

So we feel that it is time for a more systematic investigation of Dold spaces. In Section 2 we will recall the definition of Dold spaces and some facts about numerable covers. In Section 3 we will list a number of elementary facts about Dold spaces. Section 4 is the main part of the paper: We will study properties of simplicial Dold spaces and their realizations. We give a characterization of wellpointed connected Dold spaces and use it to derive results about the realization of maps of simplicial spaces which are dimensionwise fibrations. In Section 5 we apply these results to free algebras over topological operads. We close the paper with a section on counter examples.

Some of our results are well-known, some are known to specialists but have not appeared in print, some are new. We derive most of the well-known facts as special cases of more general results. We have tried to give references as well as possible, but we are not sure that we always found the original source.

We are indepted to A. Hatcher for bringing Example 6.1 to our attention and to J. Smrekar for suggesting the name “Dold space” and for e-mail exchange about function space properties of Dold spaces. The latter turned out to be so retractive that we did not include them in this paper. In fact, the category of Dold spaces is rather badly behaved with respect to function spaces. In particular, loop spaces of Dold spaces need not be Dold spaces (see Example 6.3).

## 2 Dold covers

In this section we recall the basic definitions and list results related to coverings.

Let  $\{a_j; j \in J\}$  denote a set of elements of  $\mathbb{R}_+ = \{x \in \mathbb{R}; x \geq 0\}$ . We define

$$\sum_{j \in J} a_j = \sup \left\{ \sum_{j \in E} a_j; \quad E \subset J \text{ finite} \right\}$$

**2.1 Definition:** A *partition of unity* on a space  $X$  is a set of maps  $\{f_j : X \rightarrow [0, 1]; j \in J\}$  such that

$$\sum_{j \in J} f_j(x) = 1 \quad \text{for all } x \in X.$$

**2.2 Definition:** Let  $X$  be a space. A subset  $A \subset X$  is called *ambiently contractible* if the inclusion  $A \rightarrow X$  is nullhomotopic.

**2.3 Definition:** Let  $\mathcal{U} = \{U_\alpha; \alpha \in A\}$  be a cover of  $X$ .

- (1)  $\mathcal{U}$  is called *locally finite* if each  $x \in X$  has a neighborhood  $V$  such that  $V \cap U_\alpha \neq \emptyset$  for only finitely many  $\alpha \in A$ .
- (2) A *numeration of  $\mathcal{U}$*  is a partition of unity  $\{f_\alpha; \alpha \in A\}$  of  $X$  such that  $\{\text{Supp}(f_\alpha); \alpha \in A\}$  is locally finite and  $\text{Supp}(f_\alpha) \subset U_\alpha$  for all  $\alpha \in A$ . (Recall, the support  $\text{Supp}(f)$  of a map  $f : X \rightarrow I$  is the closure of  $f^{-1}([0, 1])$ .) If  $\mathcal{U}$  admits a numeration, it is called a *numerable cover*.
- (3)  $\mathcal{U}$  is called an *ambiently contractible cover* if each  $U_\alpha$  is ambiently contractible.
- (4)  $\mathcal{U}$  is called a *Dold cover* if it is numerable and ambiently contractible.

**2.4 Definition:** A space  $X$  is called *ambiently locally contractible* if it has an ambiently contractible open cover. We call  $X$  *numerably contractible* or *Dold space* if it has a Dold cover.

**Remark:** In the literature the term “weakly contractible” is used for what we call “ambiently contractible”.

**2.5 Examples:** (1) Each discrete space is a Dold space.

(2) Each contractible space is a Dold space.

(3) Each paracompact ambiently locally contractible space is a Dold space.

(4) By [7, Thm. II. 3] each paracompact *LEC* space is a Dold space.

(5) Each *CW*-complex is a Dold space (see 3.8). The converse does not hold (see 6.1).

We will make use of the following results.

**2.6 Lemma** [3, p. 347]: If  $\{f_j, j \in J\}$  is a partition of unity on  $X$ , then  $\{f_j^{-1}(]0, 1]); j \in J\}$  is a numerable cover of  $X$ .

**2.7 Lemma** [3, p. 349]: Let  $\{f_j : X \rightarrow \mathbb{R}_+; j \in J\}$  be a set of maps such that  $\mathcal{U} = \{f_j^{-1}(]0, \infty]); j \in J\}$  is a locally finite cover of  $X$ , then  $\mathcal{U}$  is a numerable cover.

**2.8 Corollary:** The following are equivalent

(1)  $X$  is a Dold space

(2)  $X$  has a partition of unity  $\{f_j; j \in J\}$  such that  $\{f_j^{-1}(]0, 1]); j \in J\}$  is an ambiently contractible open cover of  $X$ .

(3) there is a set of maps  $\{f_j : X \rightarrow \mathbb{R}_+; j \in J\}$  such that  $\{f_j^{-1}(]0, \infty]); j \in J\}$  is a locally finite ambiently contractible cover of  $X$ .

**2.9 Proposition:** Let  $\mathcal{U} = \{U_\alpha; \alpha \in A\}$  be a cover of  $X$  by Dold spaces. Suppose that  $\mathcal{U}$  has a numerable refinement  $\mathcal{V} = \{V_j; j \in J\}$ , i.e. a numerable cover of  $X$  such that each  $V_j$  is contained in some  $U_\alpha$ . Then  $X$  is a Dold space.

**Proof:** Let  $\{f_j; j \in J\}$  be a numeration of  $\mathcal{V}$ . Since each  $U_\alpha$  is a Dold space there are partitions of unity

$$\{g_{\alpha,k} : U_\alpha \rightarrow I; k \in K_\alpha\}$$

such that  $\{\text{Supp}(g_{\alpha,k}); k \in K_\alpha\}$  is locally finite and  $\text{Supp}(g_{\alpha,k})$  is contractible in  $U_\alpha$  and hence in  $X$  for all  $k \in K_\alpha$ .

Choose a function  $\beta : J \rightarrow A$  such that  $V_j \subset U_{\beta(j)}$ . For  $k \in K_{\beta(j)}$  define  $f_{j,k} : X \rightarrow [0, 1]$  by

$$f_{j,k}(x) = \begin{cases} f_j(x) \cdot g_{\beta(j),k}(x) & \text{for } x \in \text{Supp}(f_j) \\ 0 & \text{for } x \in X \setminus f_j^{-1}([0, 1]) \end{cases}$$

Then  $f_{j,k}$  is well-defined and continuous, because  $\text{Supp}(f_j)$  and  $X \setminus f_j^{-1}([0, 1])$  are closed in  $X$ . The collection  $\{f_{j,k}; j \in J, k \in K_{\beta(j)}\}$  is a partition of unity and

$$f_{j,k}^{-1}([0, 1]) = f_j^{-1}([0, 1]) \cap g_{\beta(j),k}([0, 1]) \subset \text{Supp}(g_{\beta(j),k}).$$

Hence  $f_{j,k}^{-1}([0, 1])$  is contractible in  $X$ . Now apply 2.8(2).  $\square$

**Remark:** The numeration condition on the cover  $\mathcal{U}$  in 2.9 is essential as the following example shows.

**2.10 Examples:** Let  $X \subset \mathbb{R}^2$  be the cone on  $M = \{(0, 0)\} \cup \{(\frac{1}{n}, 0); n \in \mathbb{N} \setminus \{0\}\}$  with cone point  $(0, 1)$ . Then  $X$  is a Dold space. Now let

$$Y = (X \sqcup X)/(0, 0) \sim (0, 0).$$

The two copies of  $X$  form a closed cover of  $Y$ , but  $Y$  is not a Dold space, because no open neighborhood of  $(0, 0)$  is contractible in  $Y$ .

### 3 Elementary properties

Suppose  $U \subset X$  is ambiently contractible in  $X$  to a point  $x_0$ , then  $U$  must lie in the path-component of  $x_0$ . We obtain

**3.1 Proposition:** (1)  $X$  is a Dold space iff its path-components are open and Dold spaces.

(2) If  $X = \coprod_{j \in J} X_j$ , then  $X$  is a Dold space iff each summand  $X_j$  is a Dold space.

This observation allows us to restrict our attention to path-connected Dold spaces.

**3.2 Proposition:** A space  $Y$  dominated by a Dold space  $X$  is itself a Dold space.

**Proof:** ([4, p. 235] Let  $\{V_\lambda; \lambda \in \Lambda\}$  be a Dold cover of  $X$ , and  $f : X \rightarrow Y$  and  $g : Y \rightarrow X$  be maps such that  $f \circ g \simeq \text{id}_Y$ . Then  $\{g^{-1}(V_\lambda); \lambda \in \Lambda\}$  is a numerable cover of  $Y$  and each  $g^{-1}(V_\lambda)$  is contractible in  $Y$  because

$$g^{-1}(V_\lambda) \xrightarrow{g} V_\lambda \subset X \xrightarrow{f} Y$$

is nullhomotopic and homotopic to the inclusion  $g^{-1}(V_\lambda) \subset Y$ .  $\square$

**3.3 Corollary:** If  $X$  and  $Y$  are homotopy equivalent then  $X$  is a Dold space iff  $Y$  is a Dold space.

**3.4 Proposition:** Given a diagram

$$X \xleftarrow{f} A \xrightarrow{g} Y$$

with  $X$  and  $Y$  Dold spaces, then the double mapping cylinder  $\widehat{M}(f, g)$  is a Dold space.

**Proof:**  $\widehat{M}(f, g) = (X \sqcup A \times [0, 1] \sqcup Y) / \sim$  with  $(a, 0) \sim f(a)$  and  $(a, 1) \sim g(a)$ . Define  $\alpha : \widehat{M}(f, g) \rightarrow [0, 1]$  by

$$\alpha(z) = \begin{cases} 0 & z \in X \\ t & z = (a, t) \in A \times [0, 1] \\ 1 & z \in Y \end{cases}$$

and  $\beta : \widehat{M}(f, g) \rightarrow [0, 1]$  by  $\beta(z) = 1 - \alpha(z)$ . Then  $\{\alpha, \beta\}$  is a numerable cover,  $\alpha^{-1}([0, 1]) \simeq Y$  and  $\beta^{-1}([0, 1]) \simeq X$ . Hence  $\widehat{M}(f, g)$  is a Dold space by 2.9.  $\square$

Recall that a map  $f : A \rightarrow X$  is an *h-cofibration* if there is a commutative triangle

$$\begin{array}{ccc} & A & \\ f \swarrow & & \searrow j \\ X & \xrightarrow{h} & Y \end{array}$$

with  $j$  a cofibration and  $h$  a homotopy equivalence under  $A$ . Dually, an *h-fibration* is a map  $f : A \rightarrow X$  which is homotopy equivalent over  $X$  to a fibration.



**3.5 Corollary:** Let

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ g \downarrow & & \downarrow \\ C & \longrightarrow & X \end{array}$$

be a pushout square with  $f$  an  $h$ -cofibration and  $B$  and  $C$  Dold spaces. Then  $X$  is a Dold space.

**Proof:** Since  $f$  is an  $h$ -cofibration, the canonical map  $\widehat{M}(f, g) \rightarrow X$  is a homotopy equivalence.  $\square$

**3.6 Corollary:** (1) The unreduced suspension  $\widehat{\Sigma}X$  of any space  $X$  is a Dold space.

(2) [13, Lemma 1.3] For any map  $f : A \rightarrow X$  into a Dold space  $X$ , the unreduced mapping cone is a Dold space.

(3) If  $f : A \rightarrow X$  is an  $h$ -cofibration and  $X$  a Dold space, then  $X/f(A)$  is a Dold space.

**3.7 Proposition:** [13, Lemma 1.6] Let  $X_0 \xrightarrow{f_0} X_1 \xrightarrow{f_1} X_2 \xrightarrow{f_2} \dots$  be a sequence of maps of Dold spaces. Then

(1) the mapping telescope  $TX = (\coprod_{n \geq 0} X_n \times I) / \sim$  with  $(x, 1) \in X_n \times I$  related to  $(f_n(x), 0) \in X_{n+1} \times I$  is a Dold space.

(2) if each  $f_i$  is an  $h$ -cofibration,  $\text{colim } X_n$  is a Dold space.

**Proof:**  $TX$  is the double mapping cylinder of

$$\coprod_{n \text{ even}} X_n \xleftarrow{(g_n)} \coprod_{n \leq 0} X_n \xrightarrow{(h_n)} \coprod_{n \text{ odd}} X_n$$

$$\text{with } g_n(x) = \begin{cases} x & n \text{ even} \\ f_n(x) & n \text{ odd} \end{cases} \quad h_n(x) = \begin{cases} f_n(x) & n \text{ even} \\ x & n \text{ odd} \end{cases}$$

If all the  $f_n$  are  $h$ -cofibrations, the canonical maps  $TX \rightarrow \text{colim } X_n$  is homotopy equivalence.  $\square$

**3.8 Corollary:** [5, Prop. 6.7] Each  $CW$ -complex  $X$  and hence each space of the homotopy type of  $CW$ -complex is a Dold space.

**Proof:** Let  $X^{(n)}$  denote the  $n$ -skeleton of  $X$ . Then  $X^{(n)}$  is a Dold space by induction on  $n$  using 3.5. Hence  $X$  is a Dold space by 3.7.2.  $\square$

**3.9 Proposition:** Let  $p : E \rightarrow B$  be any map. Assume that  $B$  and the homotopy fibers  $F_b(p)$  of  $p$  over  $b$  are Dold spaces for all  $b \in B$ , then  $E$  is a Dold space.

**Proof:** By 3.1 we may assume that  $B$  is path-connected, and by 3.3 we may assume that  $p : E \rightarrow B$  is a fibration whose fiber  $F$  over a fixed  $b_0 \in B$  is a Dold space. Let  $\mathcal{U} = \{U_\lambda; \lambda \in \Lambda\}$  be an open Dold cover of  $B$  and  $\{f_\lambda : B \rightarrow [0, 1]; \lambda \in \Lambda\}$  a numeration of  $\mathcal{U}$ , and let  $\mathcal{V} = \{V_\gamma, \gamma \in \Gamma\}$  with  $\{g_\gamma : F \rightarrow [0, 1]; \gamma \in \Gamma\}$  be the corresponding data for  $F$ . Let

$$H_\lambda : U_\lambda \times I \rightarrow B$$

be a homotopy from the inclusion  $i_\lambda : U_\lambda \subset B$  to the constant map to  $b_0$ . Since  $p$  is a fibration there is a homotopy  $K_\lambda$

$$\begin{array}{ccc} p^{-1}(U_\lambda) \times 0 & \xrightarrow{\quad} & E \\ \cap & \nearrow K_\lambda & \downarrow p \\ p^{-1}(U_\lambda) \times I & \xrightarrow{H_\lambda \circ (p \times \text{id})} & B \end{array}$$

from the inclusion  $j_\lambda : p^{-1}(U_\lambda) \subset E$  to a map  $p^{-1}(U_\lambda) \xrightarrow{k_\lambda} F \subset E$ . Define maps  $\tau_{\lambda, \gamma} : E \rightarrow [0, 1]$  by

$$\tau_{\lambda, \gamma}(e) = \begin{cases} f_\lambda(p(e)) \cdot g_\gamma(k_\lambda(e)) & e \in p^{-1}(\text{Supp}(f_\lambda)) \\ 0 & e \notin p^{-1}(f_\lambda^{-1}([0, 1])) \end{cases}.$$

Since the  $k_\lambda^{-1}(g_\gamma^{-1}([0, 1]))$ ,  $\gamma \in \Gamma$ , cover  $p^{-1}(U_\lambda)$  and since the  $p^{-1}(f_\lambda^{-1}([0, 1])) \subset p^{-1}(U_\lambda)$ ,  $\lambda \in \Lambda$ , cover  $E$ ,

$$\mathcal{W} = \{W_{\lambda, \gamma} = \tau_{\lambda, \gamma}^{-1}([0, 1]); \lambda \in \Lambda, \gamma \in \Gamma\}$$

covers  $E$ . This cover is locally finite and ambiently contractible:  $K_\lambda$  deforms  $W_{\lambda, \gamma}$  into  $k_\lambda(W_{\lambda, \gamma}) \subset V_\gamma \subset F$ , and  $V_\gamma$  is ambiently contractible in  $F$ . Let  $e \in E$ . Then there is an open neighborhood  $U$  of  $p(e)$  such that  $U \cap U_\lambda = \emptyset$  for all but finitely many  $\lambda_1, \dots, \lambda_n$ . So  $p^{-1}(U)$  only meets  $p^{-1}(U_{\lambda_i})$ ,  $i = 1, \dots, n$ . Each  $k_{\lambda_i}(e)$  has an open neighborhood  $V_i$  such that  $V_i \cap V_\gamma = \emptyset$  for all but finitely many  $\gamma_{i1}, \dots, \gamma_{ir_i}$ . Then  $p^{-1}(U) \cap \bigcap_{i=1}^n k_{\lambda_i}^{-1}(V_i) \cap \tau_{\lambda, \gamma}^{-1}([0, 1]) \neq \emptyset$  only if  $(\lambda, \gamma) \in \{(\lambda_i, \gamma_{ij}); 1 \leq i \leq n, 1 \leq j \leq r_i\}$ .

So  $E$  is a Dold space by 2.8.  $\square$

**3.10 Corollary:** [13, Lemma 1.5] If  $X$  and  $Y$  are Dold spaces so is  $X \times Y$ .  $\square$

**3.11 Corollary:** Let

$$\begin{array}{ccc} P & \xrightarrow{g} & E \\ \downarrow q & & \downarrow p \\ X & \xrightarrow{f} & B \end{array}$$

be a homotopy pullback, let  $X$  and the homotopy fibers  $F_b(p)$  of  $p$  over all  $b \in B$  be Dold spaces. Then  $P$  is a Dold space.

**Proof:** The homotopy fiber  $F_x(q)$  of  $q$  over  $x \in X$  is homotopy equivalent to  $F_{f(x)}(p)$  and hence a Dold space.  $\square$

**3.12 Corollary:** Given a diagram

$$\begin{array}{ccccc} P & \xrightarrow{\quad} & & \xrightarrow{\quad} & X \\ \downarrow & & & \swarrow & \downarrow \\ Y & \xrightarrow{\quad} & Q & \xrightarrow{\quad} & B \end{array}$$

with  $B$  a path-connected Dold space, whose outer square is a homotopy pullback and whose inner square is a homotopy pushout, then  $Q$  is a Dold space.

**Proof:** If  $F(f)$  and  $F(g)$  are the homotopy fibers of  $f$  and  $g$  respectively, the homotopy fiber of the induced map  $r : Q \rightarrow B$  is homotopy equivalent to the join  $F(f) * F(g)$  (e.g. see [16, Prop. 5.5]). Since the join of two spaces is Dold space, the result follows.  $\square$

## 4 Simplicial spaces

Let  $\Delta$  denote the category of finite ordered sets  $[n] = \{0 < 1 < \dots < n\}$  and order preserving maps and  $\text{Mon } \Delta$  the subcategory of injective order preserving maps. A *simplicial space* is a functor  $X_\bullet : \Delta^{\text{op}} \rightarrow \text{Top}$ ,  $[n] \mapsto X_n$ , a *semisimplicial space* is a functor  $X_\bullet : (\text{Mon } \Delta)^{\text{op}} \rightarrow \text{Top}$ .

Let  $|X_\bullet|$  denote the usual topological realization, also called *slim realization* of the simplicial space  $X_\bullet$  and  $\|X_\bullet\|$  denote the realization of the semisimplicial space  $X_\bullet$ , also called *fat realization*. Since a simplicial space can be considered as a semisimplicial one it has a fat and a slim realization. A simplicial space  $X_\bullet$  is called *proper* if the inclusions  $sX_n \subset X_n$  of the subspaces of degenerate elements of  $X_n$  are cofibrations for all  $n$ .

**4.1 Proposition:** (1) If  $X_\bullet$  is a semisimplicial space such that  $X_0$  is a Dold space, then  $\|X_\bullet\|$  is a Dold space.

(2) If  $X_\bullet$  is a proper simplicial space such that  $X_0$  is a Dold space, then  $|X_\bullet|$  is a Dold space.

**Proof:** (1) (The idea of the proof is probably due to D. Puppe. We learnt it many years ago from H. Meiwes [12].)

Let  $\|X\|^{(n)}$  denote the  $n$ -skeleton of the fat realization. Since  $\|X\|^{(n)} \subset \|X\|^{(n+1)}$  is a cofibration it suffices to show that each  $\|X\|^{(n)}$  is a Dold space. Assume inductively that  $\|X\|^{(n-1)}$  is a Dold space. Recall that

$$\|X\|^{(n)} = \|X\|^{(n-1)} \cup_{X_n \times \partial \Delta^n} X_n \times \Delta^n,$$

where  $\Delta^n$  is the standard  $n$ -simplex. Choose two different points  $u_1 \neq u_2$  in the interior of  $\Delta^n$ . For a space  $Y$  let  $CY = (Y \times I)/(Y \times 0)$  be the cone on  $Y$  with cone-point  $*$ , and  $\varphi : CY \rightarrow I$  the map  $(y, t) \mapsto t$ . Define maps

$$\lambda_i : (\Delta^n, u_i) \xrightarrow{h_i} (C(\partial \Delta^n), *) \xrightarrow{\varphi} (I, 0) \quad i = 1, 2$$

by choosing based homeomorphisms  $h_i$ . The maps

$$X_n \times \Delta^n \xrightarrow{\text{proj.}} \Delta^n \xrightarrow{\lambda_i} I$$

together with the constant map to 1 on  $\|X\|^{(n-1)}$  define maps

$$f_i : \|X\|^{(n)} \rightarrow I.$$

Then  $\{f_1^{-1}(]0, 1]), f_2^{-1}(]0, 1])\}$  is a numerable cover of  $\|X\|^{(n)}$  by 2.7. The subspaces

$$f_i^{-1}(]0, 1]) = \|X\|^{(n-1)} \cup_{X_n \times \partial \Delta^n} X_n \times (\Delta^n \setminus \{u_i\}).$$

deformation retract onto  $\|X\|^{(n-1)}$ . Hence they are Dold spaces. So  $\|X\|^{(n)}$  is a Dold space by 2.9.

If  $X$  is a proper simplicial space the natural map  $\|X\| \rightarrow |X|$  is a homotopy equivalence. Hence  $|X|$  is a Dold space.  $\square$

**4.2 Corollary:** Let  $J$  be small category and  $D : J \rightarrow \text{Top}$  a diagram of Dold spaces. Then  $\text{hocolim } D$  is a Dold space.

**Proof:**  $\text{hocolim } D$  is a topological realization of the proper simplicial space

$$[n] \mapsto \coprod_{i,j \in J} J_n(i, j) \times D(i)$$

with  $J_n(i, j) = \{(\alpha_1, \dots, \alpha_n) \in (\text{mor } J)^n; \alpha_1 \circ \dots \circ \alpha_n : i \rightarrow j\}$  for  $n > 0$  and

$$J_0(i, j) = \begin{cases} \text{id} & i = j \\ \emptyset & i \neq j \end{cases}$$

Its 0-th space is  $\coprod_{j \in J} D(j)$  and hence a Dold space.  $\square$

We now consider the based case

**4.3 Definition:** We call a based space  $(X, x_0)$  *wellpointed*, if the inclusion  $\{x_0\} \subset X$  is a closed cofibration, and *h-wellpointed* if it is an h-cofibration.

The homotopy colimit of a diagram  $D$  generally will have a different homotopy type, when taken in the category of based spaces. Let  $BJ$  denote the classifying space of  $J$ . The inclusions of the basepoints define a map

$$BJ \rightarrow \text{hocolim } D$$

and the based homotopy colimit is the quotient  $(\text{hocolim } D)/BJ$ .

**4.4 Corollary:** Let  $D : J \rightarrow \text{Top}^*$  be a diagram of wellpointed Dold spaces and based maps. Then based-hocolim  $D$  is a wellpointed Dold space.

**Proof:** If  $D : J \rightarrow \text{Top}^*$  is a diagram of wellpointed spaces then  $BJ \rightarrow \text{hocolim } D$  is a closed cofibration. Apply 3.6.  $\square$

**4.5 Remark:** The condition that  $X$  be wellpointed can be achieved functorially by a *whiskering process*: For a based space  $(X, x_0)$  we define  $X_I = (X \sqcup I)/(x_0 \sim 1)$  and choose  $0 \in I$  as basepoint of  $X_I$ . The natural map  $q : X_I \rightarrow X$  mapping  $I$  to  $x_0$  is a homotopy equivalence. If  $X$  is h-wellpointed, it is even a based homotopy equivalence. We most often state our results for wellpointed spaces because the pushout-product theorem for cofibrations requires one factor to be a closed cofibration, but for constructions which are homotopy invariant in the based category the results extend to h-wellpointed spaces. An example is the following corollary:

**4.6 Corollary:** [13, Lemma 1.5]

(1) Given a diagram

$$X \xleftarrow{f} A \xrightarrow{g} Y$$

of h-wellpointed spaces and based maps with  $X$  and  $Y$  Dold spaces. Then the reduced mapping cylinder  $M(f, g)$  is a Dold space.

- (2) Let  $(X_\alpha; \alpha \in A)$  be a family of h-wellpointed Dold spaces. Then  $\bigvee_{\alpha \in A} X_\alpha$  is a Dold space.
- (3) Let  $X$  and  $Y$  be h-wellpointed Dold spaces. The  $X \wedge Y$  is a Dold space.
- (4) The reduced suspension  $SX$  of an h-wellpointed space is a Dold space.

**Remark:** Example 2.10 shows that 4.6.2 does not hold without some assumptions on the basepoints.

We next give a characterization of path-connected Dold spaces which needs some preparations. Let

$$p : E \rightarrow X$$

be a map of based spaces and  $F(p)$  the homotopy fiber of  $p$  over the basepoint  $*$ . With  $p$  we associate a map

$$q_\bullet : E_\bullet(p) \rightarrow \Omega_\bullet X$$

of simplicial spaces as follows:  $\Omega_n X \cong \text{Top}((\Delta^n, \Delta_0^n), (X, *))$  with the function space topology. Here  $\Delta_0^n$  is the 0-skeleton of  $\Delta^n$ . Boundaries and degeneracies are defined as for the singular functor. In fact,  $\Omega_\bullet(\ )$  is a topologized version of the singular functor

$$\text{Top}^* \rightarrow r \text{Top}^{\Delta^{op}}$$

from  $\text{Top}^*$  into the category of reduced simplicial spaces, i.e. simplicial spaces  $Y_\bullet$  with  $Y_0$  a point. It is right adjoint to the realization functor  $r \text{Top}^{\Delta^{op}} \rightarrow \text{Top}^*$ .

Let  $C\Delta^n$  denote the cone on  $\Delta^n$  with cone point  $c_0$ . We define

$$E_n(p) = \{(e, w) \in E \times \text{Top}((C\Delta^n, \Delta_0^n), (X, *)), w(c_0) = p(e)\}.$$

Boundaries and degeneracies are again defined by the corresponding maps of the standard simplices. Finally, we define

$$q_n : E_n(p) \rightarrow \Omega_n X, \quad (e, w) \mapsto w|_{\Delta^n}$$

Let  $L_n \subset \Delta^n$  be the union of edges joining the  $i$ -th with the  $(i+1)$ -st vertex of  $\Delta^n$ . Since  $L_n \subset \Delta^n$  is a strong deformation retract and the inclusion is a cofibration there is a fibration and homotopy equivalence

$$\Omega_n X \rightarrow \text{Top}((L_n, L_n \cap \Delta_0^n), (X, *)) \cong (\Omega X)^n.$$

Since  $C\Delta^n \cong \Delta^{n+1}$  we have a similar homotopy equivalence

$$E_n(p) \rightarrow F(p) \times (\Omega X)^n,$$

and

$$\begin{array}{ccc} E_n(p) & \longrightarrow & F(p) \times (\Omega X)^n \\ q_n \downarrow & & \downarrow \text{proj.} \\ \Omega_n X & \longrightarrow & (\Omega X)^n \end{array}$$

commutes. Keeping this in mind, it is easy to show that  $q_\bullet : E_\bullet(p) \rightarrow \Omega_\bullet X$  is a simplicial object in the category Pull, whose objects are maps and whose maps are commutative squares which are homotopy pullbacks. A result of V. Puppe [14] implies

**4.7**

$$\begin{array}{ccc} F(p) & \longrightarrow & ||E_\bullet(p)|| \\ \downarrow & & \downarrow ||q_\bullet|| \\ * & \longrightarrow & ||\Omega_\bullet X|| \end{array}$$

is a homotopy pullback. The horizontal maps are the inclusions of the 0-skeleta.

Let  $P(p) = \{(e, \alpha) \in E \times \text{Top}(I, X); \alpha(0) = p(e)\}$  be the mapping path-space of  $p$ . The maps

$$\begin{array}{ccc} E_n(p) \times \Delta^n & \longrightarrow & P(p) \\ (e, w, t) & \longmapsto & (e, \bar{w}) \end{array}$$

with  $\bar{w}(s) = w(s, t)$  for  $(s, t) \in C\Delta^n = (I \times \Delta^n)/(0 \times \Delta^n)$ , define a map

$$u : ||E_\bullet(p)|| \longrightarrow P(p).$$

The counit  $v : ||\Omega_\bullet X|| \rightarrow X$  of the adjoint pair

**4.8**

$$|| ? || : r \text{Top}^{(\text{Mon } \Delta)^{op}} \rightleftarrows \text{Top}^* : \Omega_\bullet$$

is induced by maps

$$\Omega_n X \times \Delta^n \rightarrow X, \quad (\sigma, t) \mapsto \sigma(t),$$

and we obtain a map of fiber sequences

$$\begin{array}{ccccc}
4.9 & F(p) & \longrightarrow & ||E_{\bullet}(p)|| & \xrightarrow{||q_{\bullet}||} & ||\Omega_{\bullet}X|| \\
& \downarrow \text{id} & & \downarrow u & \text{I} & \downarrow v \\
& F(p) & \longrightarrow & P(p) & \longrightarrow & X
\end{array}$$

Since  $||\Omega_{\bullet}X||$  is a Dold space by 4.1, the square I is a homotopy pullback by 1.6.

Consider the case where  $E$  is a point. Then  $P(p)$  is contractible, and so is  $||E_{\bullet}(p)||$ : Note that  $E_n(p) \cong \Omega_{n+1}X$ , so that  $E_{\bullet}(p)$  is the based path-space construction  $P\Omega_{\bullet}X$  in  $\text{Top}^{\Delta^{op}}$ . It is well-known that  $||P\Omega_{\bullet}X|| \simeq \Omega_0X = *$ . So  $u$  is a homotopy equivalence. Hence  $v$  is a homotopy equivalence by 1.4, provided  $X$  is a path-connected Dold space. We obtain

**4.10 Proposition:** A path-connected space  $X$  is a Dold space iff the counit  $||\Omega_{\bullet}X|| \rightarrow X$  of the adjoint pair 4.8 is a homotopy equivalence.  $\square$

Now let  $E$  be any based space. Since  $v$  is a homotopy equivalence provided  $X$  is a path-connected Dold space,  $u$  is a homotopy equivalence.

**4.11 Proposition:** If  $X$  is a path-connected Dold space, then for any based map  $p : E \rightarrow B$  the maps  $u$  and  $v$  of 4.9 are homotopy equivalences.

**Remark:** From 4.11 we obtain an alternative proof of Proposition 3.9. Let  $p : E \rightarrow X$  be a map,  $X$  a path-connected Dold space, and suppose the homotopy fiber  $F(p)$  is also a Dold space, then  $E$  is a Dold space: Consider

$$||E_{\bullet}(p)|| \xrightarrow{v} P(p) \xrightarrow{r} E$$

with  $r(e, \alpha) = e$ . The maps  $v$  and  $r$  are homotopy equivalences. Since  $E_0(p) = F(p)$  the space  $||E_{\bullet}(p)||$  is a Dold space by 4.1, and hence so is  $E$ .

**4.12 Proposition:** Let  $p_* : E_* \rightarrow X_*$  be a map of based semisimplicial spaces. Let  $F(p_n)$  denote the homotopy fiber of  $p_n : E_n \rightarrow X_n$ . If each  $X_n$  is a path-connected Dold space, then

$$\begin{array}{ccc}
||F(p_*)|| & \longrightarrow & ||E_*|| \\
\downarrow & & \downarrow \\
* & \longrightarrow & ||X_*||
\end{array}$$

is a homotopy pullback.



**Proof:** By the naturality of our constructions we have a commutative diagram of semisimplicial spaces

$$\begin{array}{ccccc}
F(p_*) & \longrightarrow & ||E_\bullet(p_*)|| & \longrightarrow & ||\Omega_\bullet X_*|| \\
\downarrow \text{id} & & \downarrow u_* & & \downarrow v_* \\
F(p_*) & \longrightarrow & P(p_*) & \longrightarrow & X_* \\
\downarrow \text{id} & & \downarrow r_* & & \downarrow \text{id} \\
F(p_*) & \longrightarrow & E_* & \xrightarrow{p_*} & X_*
\end{array}$$

Since the vertical maps are homotopy equivalences in each degree they induce homotopy equivalences of fat realizations. Hence it suffices to show that

$$\begin{array}{ccc}
|[k] \mapsto F(p_k)| & \longrightarrow & |[k] \mapsto ||E_\bullet(p_k)|| \\
\downarrow & & \downarrow \\
* & \longrightarrow & |[k] \mapsto ||\Omega_\bullet X_k||
\end{array}$$

is a homotopy pullback. For this we study the map

$$q_{n,k} : E_n(p_k) \rightarrow \Omega_n(p_k)$$

of bi-semisimplicial spaces. Since its total fat realization is independent of the order in which we realize, we may first realize with respect to  $k$  and obtain a map of semisimplicial spaces

$$\bar{q}_n : ||E_n(p_*)|| \rightarrow ||\Omega_n X_*||$$

**Claim:**  $\bar{q}_n$  is a semisimplicial object in Pull, i.e.

$$\begin{array}{ccc}
||E_n(p_*)|| & \xrightarrow{d^i} & ||E_{n-1}(p_*)|| \\
\bar{q}_n \downarrow & & \downarrow \bar{q}_{n-1} \\
||\Omega_n X_*|| & \xrightarrow{d^i} & ||\Omega_{n-1} X_*||
\end{array}$$

is a homotopy pullback for each  $n$ .

**Proof:** Let  $j \neq i$ . There is a strong deformation retraction of  $C\Delta^n$  to  $\Delta^n \cup_{v_j} [c_0, v_j]$ , where  $[c_0, v_j]$  is the line from the cone point  $c_0$  to the  $j$ -th vertex  $v_j$  of  $\Delta^n$ . This deformation retraction can be chosen compatibly with  $d^i$  yielding a commutative square

$$\begin{array}{ccc}
E_n(p_k) & \longrightarrow & F(p_k) \times \Omega_n X_k \\
d^i \downarrow & & \downarrow \text{id} \times d^i \\
E_{n-1}(p_k) & \longrightarrow & F(p_k) \times \Omega_{n-1} X_k
\end{array}$$

whose horizontal maps are homotopy equivalences. Since the fat realization preserves products up to homotopy, we obtain a commutative diagram

$$\begin{array}{ccc} ||E_n(p_*)|| & \longrightarrow & ||F(p_*)|| \times ||\Omega_n X_*|| \\ d^i \downarrow & & \downarrow \text{id} \times d^i \\ ||E_{n-1}(p_*)|| & \longrightarrow & ||F(p_*)|| \times ||\Omega_{n-1} X_*|| \end{array}$$

whose horizontal maps are homotopy equivalences. So it suffices to show that

$$\begin{array}{ccc} ||F(p_*)|| \times ||\Omega_n X_*|| & \xrightarrow{\text{id} \times d^i} & ||F(p_*)|| \times ||\Omega_n X_*|| \\ \downarrow \text{proj.} & & \downarrow \text{proj.} \\ ||\Omega_n X_*|| & \xrightarrow{d^i} & ||\Omega_n X_*|| \end{array}$$

is a homotopy pullback. But this is evident. This proves the claim.

We now apply V. Puppe's result [14] again: Since  $||F(p_*)||$  is the 0-skeleton of  $||[n] \mapsto ||E_n(p_*)|| ||$ , we obtain a homotopy pullback

$$\begin{array}{ccc} ||F(p_*)|| & \longrightarrow & ||E_\bullet(p_*)|| \\ \downarrow & & \downarrow \\ * & \longrightarrow & ||\Omega_\bullet X_*|| \end{array}$$

□

The basic idea of the argument of the previous proof is due to D. Puppe. He used it to show that the fat realization commutes with the loop space functor for path-connected semisimplicial Dold spaces.

**4.13 Proposition:** Let  $X_\bullet$  be a based semisimplicial space such that each  $X_n$  is a path-connected Dold space. Then there is a canonical homotopy equivalence

$$||\Omega X_\bullet|| \rightarrow \Omega ||X_\bullet||$$

In particular,  $\Omega ||X_\bullet||$  is a Dold space if  $\Omega X_0$  is a Dold space (e.g. if  $X_0$  is based contractible).

**Proof:** Let  $*$  denote the semisimplicial point. Apply 4.12 to the map  $p_\bullet : * \rightarrow X_\bullet$ . Since  $F(p_\bullet) = \Omega X_\bullet$  and  $||*||$  is contractible, the statement follows.

□

We will see that loop spaces of Dold spaces need not be Dold spaces. But we have

**4.14 Proposition:** If  $X$  is a path-connected h-wellpointed Dold space, then  $\Omega\Sigma X$  is a Dold space.

**Proof:** By Remark 4.5 we may assume that  $X$  is wellpointed. Let  $X_\bullet$  be the simplicial space which is the  $n$ -fold wedge of  $X$  in degree  $n$ . Boundaries  $d^i$  are the folding map for  $0 < i < n$  and projections for  $i = 0, n$ . Degeneracies are the obvious injections. Apply 4.12 to  $p_\bullet : * \rightarrow X_\bullet$ . Since  $X_\bullet$  is proper, we have homotopy equivalences

$$||F(p_\bullet)|| = ||\Omega X_\bullet|| \simeq \Omega ||X_\bullet|| \simeq \Omega |X_\bullet| = \Omega\Sigma X.$$

Since  $F(p_0)$  is a point,  $||F(p_\bullet)||$  is a Dold space.  $\square$

In his proof of 4.13 Puppe used the nerve  $N_\bullet\Omega_M X$  of the Moore loop space  $\Omega_M X$  of  $X$  rather than  $\Omega_\bullet X$ . In fact, there is a simplicial map

$$\alpha_\bullet : N_\bullet\Omega_M X \rightarrow \Omega_\bullet X$$

which is degreewise a homotopy equivalence inducing a homotopy equivalence [2, Appendix]

$$||\alpha_\bullet|| : ||N_\bullet\Omega_M X|| \rightarrow ||\Omega_\bullet X||.$$

We note that  $N_\bullet\Omega_M X$  is proper if  $X$  is wellpointed, because  $\Omega_M X$  is wellpointed respectively h-wellpointed if  $X$  is [4, (11.3)]. We obtain

**4.15 Corollary:** Suppose  $X$  is a wellpointed path-connected space. Then  $X$  is a Dold space iff  $v \circ |\alpha_*| : \mathcal{B}\Omega_M X \rightarrow X$  is a homotopy equivalence, where  $B$  is the classifying space functor and  $v$  is the map of 4.10.  $\square$

The following result is an extension of Proposition 4.12.

**4.16 Proposition:** Given a commutative diagram of based simplicial spaces

$$\begin{array}{ccc} A_* & \xrightarrow{f_*} & E_* \\ q_* \downarrow & & \downarrow p_* \\ B_* & \xrightarrow{g_*} & X_* \end{array}$$

which is a homotopy pullback in each dimension. If each  $B_n$  and each  $X_n$  is a path-connected Dold space, then

$$\begin{array}{ccc} ||A_*|| & \xrightarrow{||f_*||} & ||E_*|| \\ \downarrow ||q_*|| & & \downarrow ||p_*|| \\ ||B_*|| & \xrightarrow{||g_*||} & ||X_*|| \end{array}$$

is a homotopy pullback.

**Proof:** From 4.12 we obtain a diagram

$$\begin{array}{ccccc}
||F(q_*)|| & \xrightarrow{\quad} & ||A_*|| & & \\
\downarrow & \searrow & \downarrow & \searrow & \\
& & ||F(p_*)|| & \xrightarrow{\quad} & ||E_*|| \\
& & \downarrow & & \downarrow \\
* & \xrightarrow{\quad} & ||B_*|| & \searrow & \\
& \searrow & \downarrow & & \downarrow \\
& & * & \xrightarrow{\quad} & ||X_*||
\end{array}$$

whose front face ( $F$ ) and back face ( $B$ ) are homotopy pullbacks. Since the map  $F(q_*) \rightarrow F(p_*)$  is a homotopy equivalence in each dimension by assumption, its realization is a homotopy equivalence. Hence the left face ( $L$ ) is a homotopy pullback. If ( $R$ ) denotes the right face, we find that  $(B) + (R)$  is a homotopy pullback, because  $(L) + (F)$  is one. Hence ( $R$ ) is a homotopy pullback by 1.5.  $\square$

## 5 $k$ -spaces and free algebras over operads

Throughout this section we work in the category  $k\text{Top}$  of  $k$ -spaces and its based version  $k\text{Top}^*$ . Recall that  $X$  is a  $k$ -space if a subset  $U \subset X$  is open precisely if  $f^{-1}(U)$  is open for all maps  $f : C \rightarrow X$  and all compact Hausdorff spaces  $C$ . The inclusion functor  $i : k\text{Top} \subset \text{Top}$  has a right adjoint  $k : \text{Top} \rightarrow k\text{Top}$  obtained from  $X$  by declaring the subsets  $U$  satisfying the condition above as open. The counit of this adjunction

$$ik(X) \rightarrow X$$

is the identity on underlying sets. Hence the topology of  $k(X)$  is finer than the one of  $X$ , and we obtain

**5.1 Proposition:** If  $X$  is a Dold space so is  $k(X)$ .

Since  $i$  preserves colimits and  $k$  limits, we moreover have

**5.2 Proposition:** The results of the previous sections also hold in the category  $k\text{Top}$  respectively  $k\text{Top}^*$ .

We include  $k\text{Top}$  into our considerations because Theorems 1.1 and 1.2 are phrased in  $k\text{Top}^*$ .

In his proof of Theorem 1.2 Meiwes needed to show that  $\mathbb{C}_n^*(X)$  and the  $k$ -fold symmetric product  $SP_k(X)$  are Dold spaces if  $X$  is a (wellpointed for

$\mathbb{C}_n^*(X)$ ) Dold space. He did this by explicitly constructing Dold covers. We will obtain these results from more general easy to prove facts.

Let  $\mathcal{P}$  be a topological operad. We call  $\mathcal{P}$  reduced if  $\mathcal{P}(0)$  consists of a single element. If  $X$  is  $\mathcal{P}$ -space and  $\mathcal{P}$  is reduced, the single element of  $\mathcal{P}(0)$  determines a basepoint in  $X$ . Let  $\mathcal{P}\text{Top}$  be the category of  $\mathcal{P}$ -spaces. We have a forgetful functor

$$U : \mathcal{P}\text{Top} \rightarrow k\text{Top}$$

and, if  $\mathcal{P}$  is reduced,

$$U^* : \mathcal{P}\text{Top} \rightarrow k\text{Top}^*.$$

They have left adjoints

$$\mathbb{P} : k\text{Top} \rightarrow \mathcal{P}\text{Top} \quad \text{respectively} \quad \mathbb{P}^* : k\text{Top}^* \rightarrow \mathcal{P}\text{Top}$$

defined by

$$\mathbb{P}(X) = \coprod_{n=0}^{\infty} \mathcal{P}(n) \times_{\Sigma_n} X^n \quad \text{and} \quad \mathbb{P}^*(X) \left( \coprod_{n=0}^{\infty} \mathcal{P}(n) \times_{\Sigma_n} X^n \right) / \sim$$

The relation  $\sim$  in the definition of  $\mathbb{P}^*(X)$  is defined as follow: Let  $*$   $\in \mathcal{P}(0)$  denote the single element and let

$$\begin{aligned} \sigma_i : \mathcal{P}(k) &\rightarrow \mathcal{P}(k-1), & \alpha &\mapsto \alpha \circ (\text{id}_i \times * \times \text{id}_{k-i-1}) \\ s_i : X^{k-1} &\rightarrow X^k, & (x_1, \dots, x_{k-1}) &\mapsto (x_1, \dots, x_i, *, x_{i+1}, \dots, x_{k-1}) \end{aligned}$$

Then  $(\sigma_i(\alpha), x) \sim (\alpha, s_i(x))$ .

**5.3 Proposition:** Let  $\mathcal{P}$  be an operad such that each  $\mathcal{P}(n)/\Sigma_n$  is a Dold space, and let  $X \in k\text{Top}$  be a path-connected Dold space. Then  $\mathbb{P}(X)$  is a Dold space.

**5.4 Proposition:** Let  $X \in k\text{Top}^*$  be a wellpointed path-connected Dold sapce. Then  $\mathbb{P}^*(X)$  is a Dold space for each reduced operad  $\mathcal{P}$ .

The proofs make use of the following result of May [10, Thm. 12.2]:

**5.5 Proposition:** (1) Let  $X_\bullet$  be a simplicial  $k$ -space, then there is a natural homeomorphism  $|\mathbb{P}(X_\bullet)| \rightarrow \mathbb{P}(|X_\bullet|)$ .

(2) Let  $X_\bullet$  be a wellpointed simplicial  $k$ -space and  $\mathcal{P}$  a reduced operad. Then there is a natural homomorphism  $|\mathbb{P}^*(X_\bullet)| \rightarrow \mathbb{P}^*(|X_\bullet|)$ .

May proves the based case, but the proof applies verbatim also to the non-based case.

**Proof of 5.3:** Choose a basepoint  $x_0 \in X$  and let  $q : X_I \rightarrow X$  be the map of 4.5. Since  $X_I$  is a wellpointed Dold space the map

$$v \circ |\alpha_*| : |N_*\Omega_M X_I| \rightarrow X_I$$

of 4.15 is a based homotopy equivalence. By 5.5 we have a sequence of homotopy equivalences (we ignore basepoints)

$$|\mathbb{P}(N_*\Omega_M X_I)| \rightarrow \mathbb{P}(|N_*\Omega_M X_I|) \rightarrow \mathbb{P}(X_I) \rightarrow \mathbb{P}(X).$$

$N_0\Omega_M X_I$  is a single point. Hence

$$\mathbb{P}(N_0\Omega_M X_I) = \mathbb{P}(*) = \coprod_{n=0}^{\infty} \mathbb{P}(n)/\Sigma_n$$

which is a Dold space. Hence  $|\mathbb{P}(N_*\Omega_M X_I)|$  and  $\mathbb{P}(X)$  are Dold spaces by 4.1, because  $\mathbb{P}(N_*\Omega_M X_I)$  is proper.  $\square$

**Proof of 5.4:** If  $X$  is wellpointed the map  $q : X_I \rightarrow X$  is a based homotopy equivalence. We obtain a sequence of based homotopy equivalences

$$|\mathbb{P}^*(N_*\Omega_M X_I)| \rightarrow \mathbb{P}^*(|N_*\Omega_M X_I|) \rightarrow \mathbb{P}^*(X_I) \rightarrow \mathbb{P}^*(X).$$

Since  $\mathbb{P}^*(*) = *$ , all spaces are Dold spaces by 4.1, because  $\mathbb{P}^*(N_*\Omega_M X_I)$  is proper.  $\square$

**5.6 Corollary:** Let  $X \in k \text{ Top}$  be a Dold space. Then  $SP_k(X)$  is a Dold space.

**Proof:** Let  $\{X_\alpha; \alpha \in A\}$  be the set of path-components of  $X$ . Then  $SP_k(X)$  is the disjoint union of spaces

$$SP_{r_1}(X_{\alpha_1}) \times \dots \times SP_{r_q}(X_{\alpha_q}), \quad r_1 + \dots + r_q = k.$$

By 3.1 and 3.10 it suffices to prove the result for path-connected  $X$ .

Let  $\mathcal{Com}$  be the operad for commutative monoid structures, i.e.  $\mathcal{Com}(n)$  is a single point for each  $n$ . Then

$$\mathcal{Com}(X) = \coprod_n SP_n(X)$$

is a Dold space by 5.3. So  $SP_n(X)$  is a Dold space by 3.1.  $\square$

## 6 Counter examples

**6.1 Proposition:** There are Dold spaces which are not of the homotopy type of a  $CW$ -complex.

The following example was brought to our attention by A. Hatcher [8].

Let  $Y = \{\frac{1}{n}; n \in \mathbb{N}\} \cup \{0\} \subset \mathbb{R}$  and let  $\widehat{\Sigma}Y$  be the unreduced suspension of  $Y$ . Then  $\widehat{\Sigma}Y$  is a Dold space. Let  $f : \widehat{\Sigma}Y \rightarrow X$  be any map into a  $CW$ -complex. Since  $\widehat{\Sigma}Y$  is compact,  $f$  factors through a finite subcomplex  $A \subset X$ . Hence  $f_* : H_1(\widehat{\Sigma}Y) \rightarrow H_1(X)$  factors through  $H_1(A)$ . Let  $B_n$  be the unreduced suspension of  $\{0\} \cup \{\frac{1}{i}; 1 \leq i \leq n\}$ . Then  $B_n$  is a retract of  $\widehat{\Sigma}Y$  and  $H_i(B_n) \cong \mathbb{Z}^n$ . Hence  $H_1(\widehat{\Sigma}Y)$  is not finitely generated, but  $H_1A$  is. So the map  $f$  cannot be a homotopy equivalence.

**6.2 Corollary:** There are weak homotopy equivalences between Dold spaces which are not homotopy equivalences.

**Example:** Let  $g : R\widehat{\Sigma}Y \rightarrow \widehat{\Sigma}Y$  be a  $CW$ -approximation of  $\widehat{\Sigma}Y$  of the previous example. Then  $g$  is a weak equivalence but not a homotopy equivalence.

**6.3 Proposition:** The loop space of a Dold space need not be a loop space.

**Example:**  $\mathbb{Q}$  with the subspace topology of  $\mathbb{R}$  is not a Dold space. Let  $N_\bullet\mathbb{Q}$  denote the nerve of  $(\mathbb{Q}, +)$ . Since  $(\mathbb{Q}, +)$  is a topological group there is a homotopy equivalence  $\mathbb{Q} \rightarrow \Omega||N_\bullet\mathbb{Q}||$ . Hence  $\Omega||N_\bullet\mathbb{Q}||$  is not a Dold space but  $||N_\bullet\mathbb{Q}||$  is one.  $\square$

Recall that a *closed class*  $\mathcal{C}$  in the sense of Dror Farjoun is a full subcategory of the category  $S_*$  of wellpointed spaces of the homotopy type of a  $CW$ -complex which is closed under homotopy equivalences and based homotopy colimits [6, D1].

The class of wellpointed Dold spaces is closed under homotopy equivalences and based homotopy colimits, but Dror Farjoun's results do not generalize to this class.

**6.4 Example:** Let  $F \rightarrow E \rightarrow B$  be a fibration sequence with path-connected  $B$ . If  $F$  and  $E$  are in a closed class  $\mathcal{C}$ , then so is  $B$  [6, D. 11].

This does not hold for Dold spaces. Consider the based path-space fibration

$$\Omega C \rightarrow PC \rightarrow C$$

over the polish circle  $C$  with a nice point  $c_0 \in C$ . It is well-known that  $\Omega C$  and  $PC$  are contractible and hence Dold spaces, but  $C$  is not a Dold space.

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